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On k-Transversals

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A *k-transversal* of a family of sets $\mathscr{H}$ is a function $g: \mathscr{H} \to [\bigcup \mathscr{H}]^{k+1}$ satisfying $|\bigcup g[\mathscr{F}]| \geqslant |\mathscr{F}| + k$ for every nonempty subfamily $\mathscr{F}$ of $\mathscr{H}$ and $g(H) \subseteq H$ for every $H \in \mathscr{H}$. It is proved that a (possibly infinite) family $\mathscr{H}$ has a k-transversal if and only if the family $(H \backslash S: H \in \mathscr{H})$ has a transversal for every $S \in [\bigcup \mathscr{H}]^k$. One corollary is that a family of infinite sets has a k-transversal if and only if it has a transversal. It is also shown that if $l \leqslant k$ and $f(H) \in [H]^l$ for each $H \in \mathscr{H}$ and $\mathscr{H}$ has a k-transversal, then it has a k-transversal g such that $g(H) \supseteq f(H)$ for every $H \in \mathscr{H}$. © 1987 Academic Press, Inc.

I. INTRODUCTION AND STATEMENT OF THE RESULTS

Throughout this paper $\mathscr{H}$ denotes a (possibly infinite) nonempty family of sets and V denotes $\bigcup \mathscr{H}$. Also, k denotes any natural number. A function $g: \mathscr{H} \to [V]^{k+1}$ (the set of subsets of V of size $k+1$) is called a *k-transversal* if $|\bigcup g[\mathscr{F}]| \geqslant |\mathscr{F}| + k$ for every nonempty subfamily $\mathscr{F}$ of $\mathscr{H}$ and $g(H) \subseteq H$ for every $H \in \mathscr{H}$. This is a generalisation of the notion of a "transversal," which is just a 0-transversal. (However, "transversals" have below distinct notation in that they are regarded as functions into V rather than into $[V]^1$). It was introduced by Lovász [5], who proved the following extension of Hall's theorem [3]: a finite family $\mathscr{H}$ has a k-transversal if and only if $|\bigcup \mathscr{F}| \geqslant |\mathscr{F}| + k$ for every nonempty subfamily $\mathscr{F}$ of $\mathscr{H}$. The main result of this paper is one which clearly implies Lovász' characterisation, and which is valid also for infinite families. For any subset S of V denote by $\mathscr{H} - S$ the family $(H \backslash S: H \in \mathscr{H})$.

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THEOREM 1. *$\mathscr{H}$ has a k-transversal if and only if $|V| \geqslant k$ and $\mathscr{H} - S$ has a transversal for every $S \in [V]^k$.*

A *string* f in V is an injection from some ordinal $\alpha = \operatorname{dom} f$ into V. For $A \subseteq V$ define $D(A) = (H \in \mathscr{H}: H \subseteq A)$. For any set T write $\|T\| = |T|$ if T is finite and $\|T\| = \infty$ if T is infinite. We adopt the conventions $\infty - \infty = \infty$, and $r + \infty = \infty$, $r - \infty = -\infty$ for any integer r. Nash-Williams [6] defined the following function q on strings in V:

$$q(\varnothing) = -\|D(\varnothing)\|;$$

if $\alpha = \operatorname{dom} f = \beta + 1$ then

$$q(f) = q(f \restriction \beta) + 1 - \|D(f[\alpha]) \backslash D(f[\beta])\|;$$

if α is a limit ordinal then

$$q(f) = \liminf_{\beta < \alpha} q(f \restriction \beta) - \left\| D(f[\alpha]) \Big\backslash \bigcup_{\beta < \alpha} D(f[\beta]) \right\|.$$

It is easy to show that if $\mathscr{H}$ has a transversal then $q(f) \geqslant 0$ holds for every string f. Nash-Williams proved that for countable $\mathscr{H}$ this condition implies the existence of a transversal. In [1] this was shown to hold also if $\mathscr{H} \backslash \mathscr{H}'$ has a transversal for some countable subfamily $\mathscr{H}'$. As a corollary of Theorem 1 there follows:

COROLLARY 1a. *The family $\mathscr{H}$ with $|V| \geqslant k$ has a k-transversal if and only if it has a transversal and $q(f) \geqslant k$ for every string f in V with $|\operatorname{dom} f| \geqslant k$.*

A subfamily $\mathscr{C}$ of $\mathscr{H}$ is *critical* if it has a transversal and $g[\mathscr{C}] = \bigcup \mathscr{C}$ for every transversal g of $\mathscr{C}$. Podewski and Steffens [7] proved that if $\mathscr{H}$ is countable, then it has a transversal if and only if it does not contain a critical subfamily $\mathscr{C}$, such that $H \subseteq \bigcup \mathscr{C}$ for some $H \in \mathscr{H} \backslash \mathscr{C}$.

Another consequence of Theorem 1 is

COROLLARY 1b. *$\mathscr{H}$ has a 1-transversal if and only if it has a transversal and it contains no nonempty critical subfamily.*

For general k the corresponding result is

COROLLARY 1c. *$\mathscr{H}$ has a k-transversal if and only if it has a transversal and it does not contain a subfamily $\mathscr{C}$ such that $|\bigcup \mathscr{C} \backslash f[\mathscr{C}]| < k$ for every transversal f of $\mathscr{C}$.*

One of the main tools for the proof of Theorem 1 is a result which is of interest in itself:

THEOREM 2. *If $l \leqslant k$ and $f: \mathscr{H} \to [V]^l$ satisfies $f(H) \subseteq H$ and $\mathscr{H}$ has a k-transversal, then $\mathscr{H}$ has a k-transversal g such that $g(H) \supseteq f(H)$ for every $H \in \mathscr{H}$.*

A family $\mathscr{H}$ is *two-colorable* if there exists a coloring of V in two colors such that every $H \in \mathscr{H}$ contains vertices of both colors. If $\mathscr{H}$ has a 1-transversal, then it is two-colorable, since a 1-transversal can be considered as a forest (its 2-sets being taken as edges of a graph on V); and a forest is clearly 2-colorable. In [4] it was proved that if all members of $\mathscr{H}$ are infinite and $\mathscr{H}$ has a transversal, then it is two-colorable. Here we strengthen this to

THEOREM 3. *If all members of $\mathscr{H}$ are infinite then $\mathscr{H}$ has a k-transversal if and only if it has a transversal.*

Theorem 3 was suggested to us by E. Milner, and it was the attempt to prove it which started the investigations of this paper.

II. PROOFS

LEMMA 1. *Suppose that $k > 0$ and that g is a k-transversal of $\mathscr{H}$. If $F \in \mathscr{H}$ and $x \in F \backslash g(F)$ and a and b are distinct members of $g(F)$ then for $y = a$ or for $y = b$ there holds: the function g' defined by $g'(F) = g(F) \backslash \{y\} \cup \{x\}$ and $g'(I) = g(I)$ for $I \neq F$ is a k-transversal.*

Proof. Suppose that neither a nor b can serve in the role of y in the lemma. Then there exist finite subfamilies $\mathscr{I}$ and $\mathscr{J}$ of $\mathscr{H} \backslash \{F\}$ such that

$$|\bigcup g[\mathscr{I}] \cup (g(F) \backslash \{a\} \cup \{x\})| < |\mathscr{I}| + 1 + k \tag{1}$$

and

$$|\bigcup g[\mathscr{J}] \cup (g(F) \backslash \{b\} \cup \{x\})| < |\mathscr{J}| + 1 + k. \tag{2}$$

Let $\mathscr{K} = \mathscr{I} \cup \mathscr{J} \cup \{F\}$. Suppose first that $\mathscr{I} \cap \mathscr{J} \neq \varnothing$. Then, since g is a k-transversal, $|\bigcup g[\mathscr{I} \cap \mathscr{J}]| \geqslant |\mathscr{I} \cap \mathscr{J}| + k$. By (1) and (2) this entails

$$|\bigcup g[\mathscr{K}]| \leqslant |\mathscr{I}| + k + |\mathscr{J}| + k - (|\mathscr{I} \cap \mathscr{J}| + k) = |\mathscr{K}| - 1 + k,$$

contradicting the fact that g is a k-transversal. Thus we may assume that $\mathscr{I} \cap \mathscr{J} = \varnothing$.

Since $|g(F)| = k + 1$ we have

$$|(\bigcup g[\mathscr{I}] \cup (g(F) \backslash \{a\} \cup \{x\})) \cap (\bigcup g[\mathscr{J}] \cup (g(F) \backslash \{b\} \cup \{x\}))| \geqslant |g(F) \backslash \{a, b\} \cup \{x\}| = k.$$

Therefore, by (1) and (2),

$$|\bigcup g[\mathscr{K}]| \leqslant |\mathscr{I}| + k + |\mathscr{J}| + k - k = |\mathscr{K}| - 1 + k,$$

again contradicting the fact that g is a k-transversal.

COROLLARY. *If g is a k-transversal of $\mathscr{H}$ and $H \in \mathscr{H}$ and F is a subset of H and $|F| \leqslant k$, then there exists a k-transversal $\bar{g}$ of $\mathscr{H}$ such that $\bar{g}(H) \supseteq F$ and $\bar{g}(I) = g(I)$ for each $I \neq H$.*

Proof. It suffices to show that if $F \not\subseteq g(H)$ then it is possible to replace $g(H)$ by $g(H) \setminus \{y\} \cup \{x\}$ for some $y \in g(H) \setminus F$ and $x \in F \setminus g(H)$ while keeping g a k-transversal. But if $F \not\subseteq g(H)$ then there exists $x \in F \setminus g(H)$ and since $|F| \leqslant |g(H)| - 1$ there exist distinct a, $b \in g(H) \setminus F$, and one can apply the lemma.

The proof of Theorem 2 now follows easily. Let h be a k-transversal of $\mathscr{H}$. We well order $\mathscr{H}$ as $(H_\theta : \theta < |\mathscr{H}|)$, and one by one replace $h(H_\theta)$ by a $k+1$-subset of H_θ which contains $f(H_\theta)$, while keeping h a k-transversal. The resulting function g is clearly a k-transversal, since it suffices to check the condition $|\bigcup g[\mathscr{F}]| \geqslant |\mathscr{F}| + k$ for finite subfamilies of $\mathscr{H}$ (and this is also the reason why in limit steps of the definition of g we are still left with a k-transversal).

We shall need Theorem 2 only in the case that f is a transversal.

LEMMA 2. *If $\mathscr{H}$ has a transversal f and it can be decomposed as $\mathscr{H} = \bigcup_{\alpha < \zeta} \mathscr{H}_\alpha$ so that each $\mathscr{H}_\alpha$ has a k-transversal and $f[\mathscr{H}_\alpha] \cap \bigcup \mathscr{H}_\beta = \varnothing$ whenever $\alpha > \beta$, then it has a k-transversal.*

Proof. By Theorem 2 each $\mathscr{H}_\alpha$ has a k-transversal g_α such that $f(H) \in g_\alpha(H)$ for $H \in \mathscr{H}_\alpha$. Define: $g = \bigcup_{\alpha < \zeta} g_\alpha$. Let $\mathscr{G}$ be a finite subfamily of $\mathscr{H}$. Let α be the first ordinal such that $\mathscr{G} \cap \mathscr{H}_\alpha \neq \varnothing$. Then

$$\begin{aligned} |\bigcup g[\mathscr{G}]| &\geqslant |\bigcup g_\alpha[\mathscr{G} \cap \mathscr{H}_\alpha]| + |f[\mathscr{G} \setminus \mathscr{H}_\alpha]| \\ &\geqslant |\mathscr{G} \cap \mathscr{H}_\alpha| + k + |\mathscr{G} \setminus \mathscr{H}_\alpha| = |\mathscr{G}| + k. \end{aligned}$$

Proof of Theorem 1. Assume first that $\mathscr{H}$ has a k-transversal g. Let $S \subseteq V$ satisfy $|S| \leqslant k$. For each $H \in \mathscr{H}$ define $g'(H) = g(H) \setminus S$. Then $|\bigcup g'[\mathscr{F}]| \geqslant |\mathscr{F}|$ for any $\mathscr{F} \subseteq \mathscr{H}$. Since the sets $g'(H)$ are finite, it follows by Hall's theorem [2] that the family $(g'(H) : H \in \mathscr{H})$ has a transversal. Since $g'(H) \subseteq H \setminus S$, so does $\mathscr{H} - S$.

For the proof of the reverse direction of Theorem 1, we need first the following strengthening of the finite case of the theorem:

LEMMA 3. *Let $\mathscr{H}=(H_0,\dots,H_n)$ be a finite family. For $0\leqslant i\leqslant n$ write $\mathscr{H}_i=(H_j:j\geqslant i)$. If for each $0\leqslant i\leqslant n$ there exists a set $A_i\in[H_i]^k$ such that $\mathscr{H}_i-A_i$ has a transversal f_i, then $\mathscr{H}$ has a k-transversal.*

Proof. By induction on n. For $n=0$ the lemma is clear. Let now $n>0$. By the induction hypothesis there exists a k-transversal g of $\mathscr{H}_1$. By Theorem 2 we may assume that $f_0(H_i)\in g(H_i)$ for $1\leqslant i\leqslant n$. Extend g to a function on $\mathscr{H}$ by defining $g(H_0)=A_0\cup\{f_0(H_0)\}$. Applying Lemma 2 with f, $\mathscr{H}$, ζ, $\mathscr{H}_1$, $\mathscr{H}_0$ replaced respectively by f^*, $(g(H_i)\colon 0\leqslant i\leqslant n)$, 2, $(g(H_i)\colon 1\leqslant i\leqslant n)$ and the family whose sole member is $g(H_0)$, where $f^*(g(H_i))=f_0(H_i-A_0)$, we see that g is a k-transversal.

Lemma 3 clearly yields Theorem 1 for finite $\mathscr{H}$. Consider next the countable case, i.e., when $\mathscr{H}=(H_i\colon i<\omega)$. For each $i<\omega$ chose $A_i\in[H_i]^k$. By the assumption of the theorem there exists a transversal f_i of $\mathscr{H}-A_i$. For each $i<\omega$ let $H_i'=A_i\cup\{f_j(H_i\backslash A_j)\colon j\leqslant i\}$. For $m<\omega$ the family $(H_0',H_1',\dots,H_m')$ satisfies the conditions of Lemma 3, and hence has a k-transversal. But each H_i' is finite, and by compactness it follows that the family $(H_i'\colon i<\omega)$ has a k-transversal. Since $H_i'\subseteq H_i$, so does $\mathscr{H}$.

Consider now the case in which $\mathscr{H}$ is of general cardinality. By the assumption of the theorem, there exists a transversal f of $\mathscr{H}$. We shall show that it is possible to split $\mathscr{H}$ as

$$\mathscr{H}=\bigcup_{\alpha<\zeta}\mathscr{H}_\alpha \qquad \text{and } V \text{ as} \qquad V=\bigcup_{\alpha<\zeta}V_\alpha \quad \text{so that:}$$

(i) $\mathscr{H}_\alpha\cap\mathscr{H}_\beta=V_\alpha\cap V_\beta=\varnothing$ whenever $\alpha\neq\beta$;

(ii) $f[\mathscr{H}_\alpha]\subseteq V_\alpha$; and

(iii) each $\mathscr{H}_\alpha$ has a k-transversal contained in $\bigcup_{\beta\leqslant\alpha}V_\beta$.

By Lemma 2 it will follow then that $\mathscr{H}$ has a k-transversal.

We well order $\mathscr{H}$ as $(H_\theta\colon\theta<\mu)$. Choose $A_0\in[H_0]^k$ and let f_1 be a transversal of $\mathscr{H}$ with $f_1[\mathscr{H}]\cap A_0=\varnothing$. For any transversal h of $\mathscr{H}$ write $E(h)=\{\{H,h(H)\}\colon H\in\mathscr{H}\}$. Let Γ_1 be a bipartite graph whose sides are $\mathscr{H}$ and V, and whose edge set is $E_1=E(f)\cup E(f_1)$. Let C_1 be the connected component of Γ_1 containing H_0. Clearly C_1 is countable. Choose a set A_1 in $[V]^k$ different from A_0. Let f_2 be a transversal of $\mathscr{H}$ with $f_2[\mathscr{H}]\cap A_1=\varnothing$. Let $E_2=E_1\cup E(f_2)$ and let Γ_2 be the graph $(\mathscr{H}\cup V,E_2)$. Let C_2 be the connected component of Γ_2 containing H_0. Choose a set A_2 in $[V]^k$ different from A_0 and A_1, and let f_3 be a transversal of $\mathscr{H}$ with $f_3[\mathscr{H}]\cap A_2=\varnothing$.

Continuing in this way we obtain an ascending sequence of countable graphs C_i, $1\leqslant i<\omega$, sets $A_i\in[V]^k$, and transversals f_i of $\mathscr{H}$ with $f_i[\mathscr{H}]\cap A_i=\varnothing$, so that whenever $f(H)\in V(C_i)$ there holds $H\in V(C_{i+1})$. Note that the choice of the sets A_i has been arbitrary so far. Writing

$V_0 = V \cap \bigcup_{i<\omega} V(C_i)$, we can choose the sets A_i in such a way that each member of $[V_0]^k$ is chosen as A_i for some $i<\omega$. Let $\mathscr{H}_0 = \bigcup_{i<\omega} V(C_i) \cap \mathscr{H}$. For each $H \in \mathscr{H}_0$ let $H' = H \cap V_0$. Let $\mathscr{H}'_0 = (H' \colon H \in \mathscr{H}_0)$. By our construction $\mathscr{H}'_0 - A$ has a transversal for every $A \in [V_0]^k$. Since $\mathscr{H}'_0$ is countable, it follows that it has a k-transversal, and hence $\mathscr{H}_0$ has a k-transversal contained in V_0.

Let now $\mathscr{H}^0 = \mathscr{H} \backslash \mathscr{H}_0$, and let β_1 be the first ordinal such that $H_{\beta_1} \in \mathscr{H}^0$. Repeat the same construction with $\mathscr{H}^0$ and H_{β_1} replacing $\mathscr{H}$ and H_0, respectively. We obtain a countable subfamily $\mathscr{H}_1$ of $\mathscr{H}^0$, and a countable subset V'_i of V, such that $H_{\beta_1} \in \mathscr{H}_1$ and $\mathscr{H}_1$ has a k-transversal into V'_1. Define $V_1 = C'_1 \backslash V_0$. Repeating this process until $\mathscr{H}$ is exhausted yields the proof.

Proof of Corollary 1a. Suppose first that $\mathscr{H}$ has a k-transversal. Then, by Theorem 1, $\mathscr{H} - S$ has a transversal for every $S \in [V]^k$. Therefore $\mathscr{H}$ has a transversal. Let f be a string in V with $|\mathrm{dom} f| \geqslant k$. Let $T = \{f(0),\ldots, f(k-1)\}$ and f' be the string obtained from f by deleting its first k terms. Let q' be the function related to $\mathscr{H} - T$ as q is related to $\mathscr{H}$. Then $q'(f') \geqslant 0$ since $\mathscr{H} - T$ has a transversal, and so k applications of [6, Lemma 2.10] (whose proof remains valid for uncountable strings) yield $q(f) \geqslant k + q'(f') \geqslant k$.

Conversely, suppose that $\mathscr{H}$ has a transversal and $q(f) \geqslant k$ for each string f with $|\mathrm{dom} f| \geqslant k$. By Theorem 1 it suffices to show that $\mathscr{H}' = \mathscr{H} - A$ has a transversal for every $A \in [V]^k$. Since $\mathscr{H}'$ has a transversal, it follows that in $\mathscr{H}'$ for each string f there holds $q(f) \geqslant 0$. Let h be a transversal of $\mathscr{H}$. Then $\mathscr{H}' \backslash h^{-1}[A]$ has a transversal. By [1, Lemma 14] if a family $\mathscr{G}$ has a countable subfamily $\mathscr{F}$ such that $\mathscr{G} \backslash \mathscr{F}$ has a transversal and if in $\mathscr{G}$ there holds $q(f) \geqslant 0$ for each string f then $\mathscr{G}$ has a transversal. Thus $\mathscr{H}'$ has a transversal, completing the proof.

Proof of Corollary 1c. Suppose that $\mathscr{H}$ has a k-transversal. Then by Theorem 1 it has a transversal. Let $\mathscr{C}$ be a nonempty subfamily of $\mathscr{H}$. Since $\mathscr{H}$ has a k-transversal, it follows that $\mathscr{C}$ has a k-transversal and so, by Theorem 1, we can select a set $S \in [\bigcup \mathscr{C}]^k$ and a transversal f of $\mathscr{C}$ such that $S \cap f[\mathscr{C}] = \varnothing$, which shows that the inequality $|\bigcup \mathscr{C} \backslash f(\mathscr{C}]| < k$ is false for some transversal f of $\mathscr{C}$.

Assume that $\mathscr{H}$ possesses the property in the statement and let f be a string with $|\mathrm{dom} f| \geqslant k$. If $\mathscr{C} = \{H \in \mathscr{H} \colon H \subseteq \mathrm{rng}(f)\}$ then by [6, Lemma 1.1] $|\bigcup \mathscr{C} \backslash h[\mathscr{C}]| \leqslant q(f)$ holds for every transversal h of $\mathscr{C}$. Now we can use Corollary 1a.

Proof of Theorem 3. By Theorem 1, if $\mathscr{H}$ has a k-transversal then it has a transversal. If all members of $\mathscr{H}$ are infinite, then $q(f) \geqslant \|\mathrm{dom} f\|$ for each string f, and hence, by Corollary 1a, if $\mathscr{H}$ has a transversal then it has a k-transversal.

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REFERENCES

1. R. AHARONI, On the equivalence of two conditions for the existence of transversals, *J. Combin. Theory Ser. A* **34** (1983), 202–214.
2. M. HALL, Distinct representatives of subsets, *Bull. Amer. Math. Soc.* **54** (1948), 922–928.
3. P. HALL, On representatives of subsets, *J. London Math. Soc.* **10** (1935), 26–30.
4. G. HOFFMAN AND P. KOMJÁTH, The transversal property implies property B, *Period. Math. Hungar.* **7** (1976), 179–181.
5. L. LOVÁSZ, A generalisation of König's theorem, *Acta Math. Acad. Sci. Hungar.* **21** (1970), 443–446.
6. C. ST. J. A. NASH-WILLIAMS, Another criterion for marriage in denumerable societies, *Ann. Discrete Math.* **3** (1978), 165–179.
7. K.-P. PODEWSKI AND K. STEFFENS, Injective choice functions for countable families, *J. Combin. Theory Ser. B* **21** (1976), 40–46.